The Mathematics Teacher

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THE NEW YORK STATE REGENTS SYLLABUS IN INTERMEDIATE ALGEBRA.*

By F. F. Decker.

When your president declined my suggestion to furnish a substitute for this occasion he stated that he desired a college teacher to introduce the topic, a fact which I may seem to overlook, for I believe that in general the best course in secondary school algebra for the student who does not go to college is the best course for the one that does. I am in a position to view the product of the high school course, or perhaps I should say the better product, and I might give my conclusions as to the parts of the work that need stressing. I shall try.

I must recognize that the training the student is to acquire before he reaches me is a function of many variables, including his capacity for training at an earlier age, his interests, and his possible future. If the early training that the research mathematician, the engineer, or the statistician should have differs from that which is best for the average pupil and must be obtained in the same classroom then I fear that these types must yield to their fellow students in the determination of the curriculum. I do not believe, however, that such a conflict exists. The algebra in permanent practical use will, in general, be the best that mathematics has to offer for the purpose. The ability to use it successfully will depend on a thorough knowledge of a

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minimum number of general principles and the power to think in terms of a minimum system of symbols, in order to attack a quantitative relationship when it falls within one's experience for the first time. This suggests exactly the training that is desirable for students who are to enter college.

The backbone of the course then should be a considerable number of problems. They will not be practical problems in the sense of having the particular setting in which the student will meet them when he leaves school, for this is a matter which cannot be predicted. Nor will they be practical in the sense that he now solves them just as he will solve them later if he does encounter them. Rather they will be practical from the point of view of the course, in that they will serve as an interesting concrete introduction to the principles which we believe he should master and will provide a motive for mastering them. We should aim at only such skill in manipulation as is to be used in solving the problems, allowing liberally for leakage.

In stressing the problems we may seem to be overlooking one side of the dual nature of algebra, that of the inner relations of mathematical truths to each other—algebra as a mathematical science, with all its elegance. We must not rob the student of this heritage; but in so far as it is separable from the study of the concrete quantitative side of the student's environment, it should follow, not precede. Then he will have the fundamental ideas that will enable him to appreciate algebra as an ordered system, and an interest in it.

In the collection of sufficiently extensive lists of suitable problems many teachers will no doubt need the assistance of the State Department, extended through the syllabus or otherwise—even after we secure the benefit of the suggested plan of the requirement of a license in mathematics for the teaching of mathematics. No attempt is made in this paper to propose any definite problems. Rather I venture to suggest the scope of the algebra which these problems should involve.

The central idea of algebra seems to be the function. A very simple way to get a notion of the actual existence of functions together with their variation is, I believe, to plot them. I would introduce graphs very early in the course, the first functions plotted being preferably those arising in connection with arith-

metic; for example, percentage as a function of the base at a given rate or area of a rectangle as a function of the height with a given base, beginning with positive numbers only. Following this I would suggest the algebraic manipulations of these formulas. The graphs of given data lead to many interesting problems. If the boy takes home to his father the graphs showing the rates at which ship tonnage is being destroyed by submarines, new tonnage is being completed, and submarines are being sunk, all plotted on the same axes, the father will not be asking why the boy is studying algebra.

Long lists of exercises where the purpose is overlooked seem often to mean wasted time. For the student who advances to college mathematics the number of principles needed in algebraic manipulation is not long and the benefits accruing from much routine practice tending toward rote application of these principles is often lost in the interim before the student reaches college. If he has previously thoroughly mastered a few principles he should soon find himself able to recall them readily. He should be expected to state the principles involved in his manipulation of forms whenever asked and he should frequently be asked, in class and on the examination.

In considering the matter of dividing the work between elementary and intermediate algebra we must consider that about 77* per cent. of those who take elementary algebra leave school without taking intermediate. About 84* per cent. of those taking intermediate do not take advanced. It follows that the object of teaching elementary algebra must not be primarily that of training the student for the study of intermediate algebra as some such critics as Dr. Snedden say that it seems to be. It should be so taught as to make it thoroughly worth while for those who pursue the study of algebra no farther.

Among some teachers of elementary algebra there is a feeling that it should be a course for the development of skill in manipulation to fit the student for later courses in algebra to which all matters of reasoning should as far as possible be relegated on the ground that the student can not reason at the age of thirteen but will be able to do so later. This plan has at least two other objections in addition to the one just given. It leads

* Vide author's table in this Journal, Vol. IX., No. 2.

to its own reductio ad absurdum in that we find the teachers of college freshmen, for example, wanting to leave the reasoning for sophomores, and so on. At the age of eleven or before—at that period when the child gets interested in telegraphy, in sign languages, and the like—he seems to be getting ready for reasoning in terms of algebraic symbols, if he be but given the opportunity to get clear ideas with reference to them and to practice with them. He should start right; and then if he comes to college his freshman algebra will appeal to him as a continuation of his earlier work and not as a new subject as many students seem now to regard it. No effort has been made in the preparation of the syllabus here suggested to confine the elementary algebra to a course in manipulation based on model solutions.

Before passing to a detailed syllabus of intermediate algebra it will be necessary to outline briefly the syllabus in elementary algebra presupposed. This will be done with reference to the elementary algebra syllabus now in use.

A. The following omissions are suggested in so far as the brevity of the course demands:

1. The removal of signs of aggregation in cases where any term is affected by more than two such signs.
2. The factorization of $x^n \pm y^n$, where n is greater than 3.
3. Most of proportion except the definition and a very few problems in which the proportions are treated as other equations.
4. Most of the work of the reduction of surds and the fundamental operations on surds.
5. The square root of polynomial algebraic expressions, except by inspection.

B. The following additions are presupposed:

1. Variation, including the determination of the constant.
2. Graphs, including:
   (*a*) Graphic representation of a function;
   (*b*) Interpolation by means of the graph;
   (*c*) Graphs of given data;
   (*d*) The graph of the linear equation;

(*e*) Approximate solution by graphs of simultaneous linear equations;

(*f*) The graphic treatment of inconsistent and indeterminate pairs of linear equations;

(*g*) Solution of the quadratic equation by graphs.

I suggest the following syllabus in intermediate algebra: (The parts of the syllabus not starred are identical with parts of the current Regents syllabus in the subject.)

A. A thorough review of elementary algebra including considerable oral work.

B. Factoring

1. Trinomial form $a^4 + a^2b^2 + b^4$.

*2. Simple polynomials of the third or fourth degree by the factor theorem, together with their graphs.

*3. $x^n \pm y^n$, $n > 3$.

C. Fractions. Simple types of fractions whose numerators and denominators are themselves fractions.

D. Exponents and Radicals.

1. Proof of (*a*) $a^m . a^q = a^{m+q}$; (*b*) $a^m \div a^q = a^{m-q}$, $m > q$; (*c*) $(a^m)^q = a^{mq}$; where m and q are positive integers.

2. Meaning of negative, fractional and zero exponents.

*3. The use of these exponents, including the use of negative exponents in manipulating decimal proper fractions containing several zeros at the immediate right of the decimal point.

*4. Application of the laws of exponents to the manipulation of surds.

*5. Square root of a binomial quadratic surd by inspection, in simple cases only.

*6. Incommensurable exponents, introduced by means of the graph of $a^x (e.g., 16^x)$ and the idea of continuity.

*7. Theory of logarithms as exponents.

*8. Computation by means of a four-place logarithm table.

E. Quadratic equations both numerical and literal.

1. Solution of affected quadratics (*a*) by factoring, (*b*) by formula.

2. Relations between roots and coefficients.

3. Formation of an equation with two given roots.
*4. Number of roots (*a*) never more than two, (*b*) application of the discriminant, (*c*) graphic illustration.

F. Simultaneous equations.

1. Two homogeneous equations of the second degree.
2. Two symmetric equations, one of the third or fourth degree, readily solvable by dividing the variable member of one by the variable member of the other; *e. g.*, $x + y = 5$, $x^3 + y^3 = 35$.
*3. Points of intersection of curves, with graphic approximation.

Good training at this point of a student's work should lead him to consider the possibility of some combination of the given equations resulting in simpler forms before employing a general routine method for the special case before him. Values obtained for the variables should be properly associated in presenting written answers.

G. Binomial theorem. Application of the theorem in the case of the positive integral exponent, including the finding of the r-th term.

H. Progressions.
1. Arithmetical.
(*a*) Proof that with the usual notation

$$l = a + (n - 1)d.$$

(*b*) Proof that with the usual notation

$$S = \frac{n}{2}(a + l) = \frac{n}{2}[2a + (n - 1)d].$$

(*c*) Applications depending on these formulas.
2. Geometric.
(*a*) Proof that with the usual notation $l = ar^{n-1}$.

(*b*) Proof that with the usual notation $S = \frac{a(1 - r^n)}{1 - r}$

(*c*) Proof that with the usual notation $S = \frac{a}{1 - r}$ for an infinite series, $r < 1$.

(*d*) Applications depending on these formulas.

The modifications of the present syllabus in intermediate algebra here suggested may be briefly stated. The introduction to graphic work is moved forward to elementary algebra in order that the student may have the aid of this simple device in his first year and so that, in case he pursues the study of algebra no farther, he may not be deprived of a tool so useful and so much used today outside the schoolroom. This tool is not to be allowed to become rusty during the study of intermediate algebra.

The introduction to logarithms is placed in intermediate algebra. Consigning this topic to trigonometry has several disadvantages. Many students who carry their mathematical study through the course in trigonometry seem to get the idea that the usefulness of logarithms is confined to trigonometry. Those who do not reach trigonometry lose what is at once an important tool and an elegant system. In the separation of the treatment of exponents and logarithms the theory seems to contain a discontinuity. The laws for commensurable exponents are carefully proved and then the theory of logarithms is made to appear to rest on that of exponents. But most of the logarithms used are incommensurable! The plan here proposed is to introduce incommensurable exponents with reference to the continuity of an exponential curve and to pass immediately to logarithms.

The extraction of the square root of a binomial, one term of which is a quadratic surd, is to be omitted, except by the inspection method, if the pressure of time demands it.

The work is to be limited to the real domain. Imaginaries are to be studied with complex numbers in advanced algebra. This, I believe, will help clear up the all too common impression that complex numbers are, using the word with its popular meaning, imaginary. This plan will require the rephrasing of some of the work on quadratics, for example, when the discriminant is negative the quadratic will have no roots. This interpretation will be very clearly pictured by the lack of points of intersection on the graph with the axis of abscissas. It will correspond also to the irreducibility, for example of $x^2 + a^2$.

In the report of the American Mathematical Society furnishing a "Definition of College Entrance Requirements"* in elementary algebra, on which, we are told, the present syllabus in

* *Bulletin of the American Mathematical Society,* Vol. X, No. 2, November, 1903.

elementary and intermediate algebra is based, there is no mention of imaginary numbers and no indication as to whether the study of graphs is to be begun in elementary or intermediate algebra. On the other hand the use of the graph has been greatly extended since the publication of that report in 1903. The report itself suggests a revision at the end of perhaps ten years to meet new conditions.

A much later report, that of the Committee on the Teaching of Mathematics to Students of Engineering,† 1911, calls for the teaching of logarithms in logical connection with the subject of exponents rather than with trigonometry. It also states that the radical sign, except in the case of square roots, and sometimes in the case of cube roots, should always be replaced by fractional exponents, when it is desired to compute with these quantities, thus doing away with all special rules for the manipulations of radicals beyond the general laws of exponents. Such a treatment of radicals I have provided for.

It is not my purpose to try to tell the teacher of the secondary school just how much can be accomplished in the time alloted to the study of elementary or intermediate algebra. If the proposed syllabus seems too brief it is to be remembered that its purpose is to outline a minimum course, to which the teacher may add; while if appears too long I suggest the elimination of some part of it. I have prepared it on the hypothesis that the present syllabus outlines about the correct amount of work. What I am trying to do is to suggest some changes which I think will bring our excellent syllabus even more thoroughly into line with the best current thought in regard to secondary education.

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† "Syllabus of Mathematics," Society for the Promotion of Engineering Education, 1912.

A GEOMETRIC REPRESENTATION.*

By E. D. Roe, Jr.

(*Continued from Vol. X, page 210.*)

§ 4. The Surface on Which a Family of Spirals Lies.

Given the equation of the family of spirals

(1) $$y = F(x)e^{f(x,k)i}; \tag{12}$$

then by § 3, the equation of the surface on which the family lies is

$$y = F(x)e^{\phi i} \tag{13}$$

or $$y^2 + z^2 = (F(x))^2 \tag{14}$$

in rectangular co-ordinates, a surface of revolution.

(2) Given $y = F(x, \tan f(x,k), \operatorname{ctn} f(x,k))e^{f(x,k)i}$

$$= H(x, \tan f(x,k))e^{f(x,k)i}; \tag{15}$$

then the surface on which the family lies is

$$y = H(x, \tan\phi)e^{\phi i} \tag{16}$$

or in rectangular co-ordinates

$$y^2 + z^2 = \left(H\left(x, \frac{z}{y}\right)\right)^2. \tag{17}$$

This applied to $y = R(x, \phi)e^{\phi i}$ gives

$$y^2 + z^2 = \left(R\left(x, \frac{z}{y}\right)\right)^2, \tag{18}$$

which after reducing becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The difference between cases (1) and (2) is that $F(x)$ in (1) *may not be expressed* in the form $F(x) = H(x, \tan f(x,k))$. This will be true for every surface of revolution.

§ 5. The Length of an Arc of the Spiral on the Surface $f(x, y, z)=0$.

If $F_1(x, f(x, k))=f_1(x)$, $F_2(x, f(x, k))=f_2(x)$, an arc s of the spiral is given by

$$s=\int(1+(f_1'(x))^2+(f_2'(x))^2)^{\frac{1}{2}}\,dx+C. \qquad (19)$$

Examples.

1. The length of one turn of the helix $y=ae^{nxi}$ (n a parameter) is

$$\begin{aligned} s &= \int_0^{2\pi/n}(1+a^2n^2\sin^2 nx+a^2n^2\cos^2 nx)^{\frac{1}{2}}\,dx \\ &= \int_0^{2\pi/n}(1+a^2n^2)^{\frac{1}{2}}\,dx=\left(\frac{1}{n^2}+a^2\right)^{\frac{1}{2}}2\pi. \end{aligned} \qquad (20)$$

2. If $a=b=c$ in the ellipsoid $y=R(x, \phi)e^{\phi i}$, we get

$$y=(a^2-x^2)^{\frac{1}{2}}e^{\phi i}, \qquad (21)$$

the equation of a sphere.

$$y=(a^2-x^2)^{\frac{1}{2}}e^{kxi} \qquad (22)$$

is the equation of a particular family of spirals on the sphere where $f(x)=x$. The length of an arc of one of the spirals is

$$s=\int\left(\frac{a^2}{a^2-x^2}+k^2(a^2-x^2)\right)^{\frac{1}{2}}dx+c, \qquad (23)$$

reducible by $x=a\sin\phi$ and $k=2n\pi/a$ to

$$s=a\int(1+(2n\pi)^2\cos^4\phi)^{\frac{1}{2}}d\phi+C. \qquad (24)$$

§ 6. The Function $y=(x^{x-1}(x-1)^x)^{1/1-2x}$.

I. *Preliminary Statement.*

The following is a statement of some of the results obtained in the investigation of this function.

For any value of x, y is in general multiple-valued.

There is an infinite number of p-valued y's ($p=1, 2, 3 \cdots \infty$). When x is commensurable the number of values of y is finite, and infinite when x is incommensurable. All the values of

(x, y) for a given x are represented by points on the same circle with center in OX, perpendicular to OX, at distance x from O, and with radius equal to $|y|$. From $x = -\infty$ to $x = 0$ and from $x = 1$ to $x = \infty$ there is a real continuous curve in addition to an infinity of discrete real points. From $x = -\infty$ to $x = 0$, the continuous real values of y are negative, the discrete real values positive. From $x = 1$ to $x = \infty$ this is reversed. The axis of x is an asymptote, and for $\begin{matrix} x=0 \\ x=1 \end{matrix}$, $y = \infty$, and $\begin{matrix} x=0 \\ x=1 \end{matrix}$ are asymptotes. From $x = 0$ to $x = 1$, there is no continuous real curve but only discrete real positive and negative points and complex points. From $x = -\infty$ to $x = \infty$ there is an infinity of conjugate right- and left-handed spirals which pierce the real plane at the discrete real points or in other real points. The discrete real points lie infinitely close together, but so that between any two positive or any two negative real points however close, an infinity of both real and complex points still lies. The same is true for any two purely imaginary points. As x advances and the points representing y move on spirals the distribution of points on the circles may be best described as one of infinitely complex kaleidoscopic change. The spirals pass through the points on the circles, a point corresponding or belonging to a group of spirals, but in the case of an incommensurable x there is a one-to-one correspondence, one value belonging to one spiral and conversely. When y is single-valued it belongs to all spirals, which pass through it. From $x = 0$ to $x = \frac{1}{2}$ and from $x = 1$ to $x = \infty$ $|y|$ decreases, and from $x = \frac{1}{2}$ to $x = 1$ and from $x = -\infty$ to $x = 0$ $|y|$ increases. At $x = \frac{1}{2}$, $|y| = 2e = 5.4+$ approximately and $|y|$ is at a minimum. At $x = \infty$, $|y| = 0$, an absolute minimum. The whole representation both for all real and all complex points is symmetrical with respect to the point $(\frac{1}{2}, 0)$. The positive and negative discrete real points lie on the real curves of $y = \pm(x^{x-1}(1-x)^x)^{1/1-2x}$, the upper sign for the curve through the positive points. These real curves are symmetrical with respect to $(\frac{1}{2}, 0)$ and also with respect to the lines $y = 0$, $x = \frac{1}{2}$. When $x > 1$, these curves contain complex and discrete real points which lie on the real curves of $y = \pm(x^{x-1}(x-1)^x)^{1/1-2x}$. They also contain spirals. All

the real and complex points and spirals of all the curves mentioned lie on the surface of revolution

$$y = (x^{x-1}(1 - x)^x)^{1/(1-2x)}e^{\phi i} \tag{25}$$

or

$$y^2 + z^2 = (\mathrm{mod}\,(x^{x-1}(x - 1)^x)^{1/(1-2x)})^2. \tag{26}$$

II. *Properties and Special Values of the Function.*

1. *Symmetry.*—Transforming to (½, 0) as origin we have

$$y_x = ((x + \tfrac{1}{2})^{x-\frac{1}{2}}(x - \tfrac{1}{2})^{x+\frac{1}{2}})^{-(1/2x)}. \tag{27}$$

Changing x into $-x$, we find

$$y_{-x} = -y_x. \tag{28}$$

For the curves through the real discrete points between 0 and 1

$$y_x = \pm (x^{x-1}(1 - x)^x)^{1/(1-2x)}$$

we get for (½, 0) as origin,

$$y_{-x} = \pm y_x. \tag{29}$$

Hence the discrete real points lie on curves which are symmetrical with respect to both axes and the origin (taken at (½, 0)), while the points themselves and all points of the whole representation are symmetrical with respect to (½, 0), which is therefore a center of symmetry of the configuration.

2. *The Variation of* $|y|$.—Since $y = |(x^{x-1}(x-1)^x)^{1/1-2x}|$ we may differentiate $y = (x^{x-1}(1-x)^x)^{1/1-2x}$ to study the change in the distance from the axis of x of the distribution of points between 0 and 1. For (½, 0) as origin, we find

$$\frac{dy}{dx} = \frac{y}{4x^2}\left(\log\left(\frac{1 - 2x}{1 + 2x}\right) - \frac{4x(4x^2 + 1)}{4x^2 - 1}\right).$$

Developing,

$$\frac{dy}{dx} = \frac{y}{2x^2}\left(\frac{5}{3}(2x)^3 + \frac{9}{5}(2x)^5 + \cdots + \frac{4n + 1}{2n + 1}(2x)^{2n+1} + \cdots\right), \tag{30}$$

from which it appears that between the limits $-½$ and ½ (0 and 1)

$$\frac{dy}{dx} \gtreqless 0 \quad \text{according as} \quad \begin{matrix} 0 < x < \frac{1}{2} \\ x=0 \\ -\frac{1}{2} < x < 0 \end{matrix}. \tag{31}$$

Hence $|y|$ is decreasing from $x=0$ to $x=\frac{1}{2}$ (original origin) and increasing from $x=\frac{1}{2}$ to $x=1$. At $x=\frac{1}{2}, |y|=2e=5.4$ approximately (see 3) and this value is a minimum. Similarly it can be shown that $|y|$ increases from $-\infty$ to 0, and decreases from 1 to ∞.

3. *The Values of* $|y|$ *when* $x=\pm\infty,\ 0,\ \frac{1}{2},\ 1$.—When $x=\pm\infty$, the value of $|y|$ is easily found to be 0; when $\begin{matrix} x=0 \\ x=1, \end{matrix}$ $|y|=\infty$. These values show that $y=0$, $x=0$ and $x=1$ are asymptotes. When $x=\frac{1}{2}$ put $x=\frac{1}{2}-\alpha$ (α positive or negative). Then

$$y=\left(\frac{2}{1-2\alpha}\right)\left(1+\frac{4\alpha}{1-2\alpha}\right)^{(1-2\alpha)/4\alpha}(-1)^{(1-2\alpha)/4\alpha} \qquad (32)$$

and as $\alpha \doteq 0$, $x \doteq \frac{1}{2}$,

$$\lim_{x \doteq \frac{1}{2}} |y| = 2e. \qquad (33)$$

4. *The Value Systems Between 0 and 1.**—The value systems between 0 and 1 are of especial interest as here y may have both positive and negative discrete values, but not for the same value of x. Elsewhere this can frequently occur. In fact as necessary for both positive and negative discrete real values, either

$$\frac{x-1}{1-2x}=\pm\frac{1}{2n},$$

whence

$$x=\frac{\pm 2n+1}{\pm 2n+2} \quad \text{and} \quad \frac{x}{1-2x}=\frac{\pm 2n+1}{\mp 2n},$$

or

$$\frac{x}{1-2x}=\pm\frac{1}{2n},$$

whence

$$x=\frac{1}{\pm 2n+1}.$$

These values of x for the upper sign lie between 0 and 1, but give complex values for y (when $n=1$ pure imaginary values). Outside of this range positive and negative real values can occur

* On account of symmetry we need to consider only values between 0 and $\frac{1}{2}$.

together. For single real values between 0 and 1 it is necessary either that

$$\frac{x}{1-2x} = \pm 2n,$$

or that

$$\frac{x}{1-2x} = \pm 2n + 1,$$

whence

$$x = \frac{\pm 2n}{\pm 2n + 1}, \quad \text{or} \quad x = \frac{\pm 2n + 1}{\pm 4n + 3},$$

which values of x for the upper sign lie between 0 and 1, while the other values lie outside of this range. Since

$$y = x^{(x-1)/(1-2x)}(x-1)^{x/(1-2x)},$$

it will contain a factor between 0 and 1.

$$\begin{aligned} \iota = (-1)^{x/(1-2x)} = \cos\left\{(2n+1)\frac{x}{1-2x}\pi\right\} \\ + i\sin\left\{(2n+1)\frac{x}{1-2x}\pi\right\} = e^{(2n+1)\frac{x}{1-2x}\pi i} \end{aligned} \tag{34}$$

(n is any positive or negative integer or zero).

$$\text{Put } (2n+1)\frac{x}{1-2x} = u^* \tag{35}$$

(where if $x < \frac{1}{2}$, $u > 0$; if $x > \frac{1}{2}$, $u < 0$) then

$$x = \frac{u}{2n + 2u + 1}. \tag{36}$$

ι will be equal to $+1$, i, -1, $-i$ according as

$$u = \frac{4p}{2}, \quad \frac{4p+1}{2}, \quad \frac{4p+2}{2}, \quad \frac{4p+3}{2} \tag{37}$$

(p a positive or negative integer or zero) and conversely, and for all other values of u, $\iota = \cos u\pi + i \sin u\pi = e^{u\pi i}$ will be a complex number of modulus 1 containing both the real and imaginary parts, and conversely, while simultaneously y will have

* $1 + (2n+1)\frac{1-2x}{x-1} = u'$ gives the conjugate point (see § 6, 5) and need not therefore be considered in detail.

a value

$$y = +|y|,\ +|y|i,\ -|y|,\ -|y|i,\ |y|e^{u\pi i} \tag{38}$$

and conversely. From (36) doubly infinitely many values of x can be found giving among them all the different forms for y. Conversely given any value of x between 0 and 1, the character of y can be stated by reference to (36) and (35). Thus from (36) according as u is of the form (37)

$$\left.\begin{array}{lll}
(1)\ x = \dfrac{4p}{2(2n+4p+1)} = \dfrac{4p}{4r+2} = \dfrac{2p}{2r+1}, & \text{and gives a} \\
 & \text{value of } y = +|y|, \\
(2)\ x = \dfrac{4p+1}{4r} & \text{and gives a} \\
 & \text{value of } y = +|y|i, \\
(3)\ x = \dfrac{4p+2}{4r+2} = \dfrac{2p+1}{2r+1}, & \text{and gives a} \\
 & \text{value of } y = -|y|, \\
(4)\ x = \dfrac{4p+3}{4r} & \text{and gives a} \\
 & \text{value of } y = -|y|i.
\end{array}\right\} \tag{39}$$

Any other forms for x give complex values only $y = |y|e^{u\pi i}$.

Examples.

1. For $x = \frac{2}{7}$, y is three valued with one real positive value.
2. $x = \frac{4}{7}$ gives $y = 4^3 3^4/7^7$. It has no other values.
3. The formula (36) yields series of given types. Thus for

$$\left.\begin{array}{l}
u = 2,\ n = 0, 1, 2, \cdots, x = \dfrac{2}{5}, \dfrac{2}{7}, \dfrac{2}{9}, \cdots, \dfrac{2}{2n+5}, \\
u = 4,\ n = 0, 1, 2, \cdots, x = \dfrac{4}{9}, \dfrac{4}{11}, \dfrac{4}{13}, \cdots, \dfrac{4}{2n+9}.
\end{array}\right\} \tag{40}$$

For these values of x we get at least one real positive value each for y and when there are other values they are not real.

$$\left.\begin{array}{l}
u = 1,\ n = 0, 1, 2, \cdots, x = \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \cdots, \dfrac{1}{2n+3}, \\
u = 3,\ n = 0, 1, 2, \cdots, x = \dfrac{1}{7}, \dfrac{1}{9}, \dfrac{1}{11}, \cdots, \dfrac{1}{2n+7}.
\end{array}\right\} \tag{41}$$

These values of x give at least a value of y real and negative.

4. If $x = 1/\sqrt{5}$, $u = (2n+1)(\sqrt{5}+2)$ is incommensurable, y is complex always and has an infinite number of values.

5. If $x = \frac{1295}{32657}$, $u = (2n+1)\frac{1295}{30067}$, y has 30067 values, of which one for $n = \frac{30067-1}{2}$ is real and negative ((3) of (39)).

6. For $x = \frac{11}{36}$, we see from ((39)(4)) that y has a value $y = -|y|i$ and there are no real values among the 14 values of y. In this case $u = (2n+1)\frac{11}{14}$; for $n = 3$, $u = \frac{11}{2} = 4 + {}^3$.

In general y is multiple-valued, having from one to an infinite number of values. This may be exhibited as follows: When $x = l/m$, and $m - 2l = \pm 1$ (l and m positive integers),

$$y = \left(\frac{l}{2l \pm 1}\right)^{-1} (-l \mp 1)^{\pm l} = \left(\frac{2l \pm 1}{l}\right)(-l \mp 1)^{\pm l}. \quad (42)$$

Here $x = l/(2l \pm 1)$; there is only one value of y which will be real and positive or negative according as l is even or odd. Thus there will be infinitely many single real positive and infinitely many single real negative values of y. y can not be single-valued when x is negative, nor therefore when x is greater than 1. In general when $x = l/(2l+p)$,

$$y = (-1)^{l/p}\left(\frac{2l+p}{l}\right)\left(1 + \frac{p}{l}\right)^{l/p}, \quad (43)$$

from which it is seen that for $x = l/(2l+p)$ and its symmetrical $x = (l+p)/(2l+p)$ when l and p are relatively prime an infinite number of values of y have p values for $p = 1, 2, 3, \cdots$ and that when p is even none of these values are real, and when it is odd one value is real and positive, or negative as l is even or odd. (Outside of 0 and 1 it is possible to have two real values for y equal and opposite in sign.) If $l = 2m$, $p = \infty$, $y = \infty$.

$$\lim_{l \doteq \infty} \frac{l+p}{2l+p} = \tfrac{1}{2}.$$

When x is incommensurable y has an infinite number of values all complex unless x lies outside of 0 and 1 when there is always one real value. All the values real and complex lie on a circle of radius $|y|$ for a value of x. All the circles lie on the surface $y = |y|e^{\phi i}$ otherwise expressed in (25) and (26).

5. *The Value Systems Between* $-\infty$ *and* 0, *and* 1 *and* ∞.—

For a negative x, $x = -a$,

$$y = -\left(\frac{1}{a}\right)^{\frac{a+1}{2a+1}}\left(\frac{1}{a+1}\right)^{\frac{a}{2a+1}}, \tag{44}$$

hence for every negative value of x there is always a real negative value of y. Also similarly to (35)

$$1 + 2m\frac{x}{1-2x} = -u, \quad \text{or} \quad 1 + 2m\frac{(x-1)}{1-2x} = -u.^{*} \tag{45}$$

$$x = \frac{-(u+1)}{2(m-u-1)}, \quad \text{or} \quad x = -\frac{(u+1-2m)}{2(m-u-1)} \tag{46}$$

(m an integer, $m-u-1>0$) give formulas from which the value systems can be studied. If $-u = -2p$ (p an integer >0),

$$x = -\frac{(2p+1)}{2(m-2p-1)}, \quad \text{or} \quad x = -\frac{(2p+1-2m)}{2(m-2p-1)}, \tag{47}$$

If in (46) $m = 0$, $u = -1$, x becomes indeterminate. If $p = 0$,

$$x = -\frac{1}{2(m-1)}, \quad \text{or} \quad x = +\frac{(2m-1)}{2(m-1)},$$

If $p > 0$, $m = 1$,

$$x = \frac{+(2p+1)}{4p}, \quad \text{or} \quad x = \frac{2p-1}{4p},$$

If $m = 2$,

$$x = -\frac{(2p+1)}{2(1-p)}, \quad \text{or} \quad x = \tfrac{1}{2}.$$

The last five values of x must be excluded since they do not satisfy $m-2p-1>0$, and give positive values for x. For

$$x = \frac{-(2p+1)}{2(m-2p-1)},$$

$$y = \left(\frac{2(m-2p-1)}{-2p-1+2m}\right)^{\frac{2p+1}{2(m+p)}}\left(\frac{2(m-2p-1)}{2p+1}\right)^{\frac{2p+1-2m}{2(m+p)}}(-1)^{\frac{m}{m+p}}. \tag{48}$$

* $1 + 2m\dfrac{(x-1)}{1-2x} = u'$ gives the conjugate point (see (62)).

If $m+p=2r+1$, y has both a positive and negative real value at least. If $p=2q$, $m=2s+1$,

$$x=-\frac{(2p+1)}{2(m-2p-1)}=\frac{-(4q+3)}{4(s-2q)}, \tag{49}$$

and if $p=2q+1$, $m=2s$,

$$x=-\frac{(4q+3)}{2(2s-2q-3)}. \tag{50}$$

For $x>1$, symmetry requires that there shall be in addition to the continuous positive real value of y also negative values for special values of x, as for

$$x=1+\frac{2p+1}{2(m-2p-1)}=\frac{2m-2p-1}{2(m-2p-1)},$$

III. *The Sequence and Contiguity of the Real Discrete Points.*

Some examples of the method of proving that between two discrete real positive or discrete real negative points, however close, an infinity of real and complex points still lies, will be given. It is only necessary to prove this of the positive discrete values between $-\infty$ and 0, and between 0 and 1. Symmetry will show that it is true of the negative discrete values between 1 and ∞ and between 0 and 1. Similar relations of contiguity exist for the complex points. From $-\infty$ to 0, positive discrete values exist for

$$x=-\frac{(2p+1)}{2(m-2p-1)}$$

when $m+p$ is odd by (48). Put $q=2p+1$, $t=m-2p-1$. Then $x=-(q/2t)$ gives a positive value for y. Then

$$x=-\frac{(lq\pm 2r)}{2lt},$$

$l=2s+1$, r an integer, gives a positive real point for y since

$$\begin{aligned} x&=-\frac{lq\pm 2r}{2lt}=-\frac{(2pl+l\pm 2r)}{2(lm-2lp-l)}\\ &=-\frac{(2(pl\pm r+s)+1)}{2((lm\pm 2r)-2(lp\pm r+s)-1)}=-\frac{(2(p'+1))}{2(m'-2p'-1)}, \end{aligned} \tag{51}$$

and $m' + p'$ is odd, if r and s are odd, or both even. Also

$$-\frac{lq+2r}{2lt} < -\frac{q}{2t} < -\frac{lq-2r}{2lt} \tag{52}$$

As $l \doteq \infty$

$$-\frac{(lq \pm 2r)}{2lt} \doteq -\frac{q}{2t}.$$

Moreover we may take $r = 1, 2, 3 \cdots$. (For every $x < 0$, there is always one real negative point and generally complex points. Incommensurable values of $x < 0$ and $x > 1$ always give one real point for y. Between 0 and 1 they give no real points.) Hence the statement for points between $-\infty$ and 0. Symmetry shows the same relation for the negative discrete points for $x > 1$.

From 0 to 1 by (36), if $u = 2p$, $q = 2n + 2u + 1$, $x = 2p/q$ and this by (1) of (39) gives a positive y. Similarly

$$x' = \frac{2lq}{lq \pm (2r+1)}, \; l = 2m, \tag{53}$$

also gives a positive y, and

$$\frac{2lp}{lq-(2r+1)} > \frac{2p}{q} > \frac{2lp}{lq+(2r+1)}. \tag{54}$$

As $l \doteq \infty$,

$$\frac{2lp}{lq \pm (2r+1)} \doteq \frac{2p}{q}, \quad \text{or} \quad x' \doteq x.$$

We might have taken

$$x' = \frac{2lp}{lq \pm 2r}$$

with l odd. Since r may be taken as $0, 1, 2, \cdots$, it follows that an infinite number of real positive points lie infinitely near to $x = 2p/q$.

In like manner

$$x = \frac{\dfrac{4p+1}{2}}{2n + 4p + 2}$$

gives a value of $y = +|y|i$.

$$x' = \frac{\dfrac{4lp \pm 1}{2}}{lq \pm (2r+1)}$$

if l is odd gives a value of $y = \pm|y|i$, where

$$\lim_{l \doteq \infty} x' = \frac{2p}{q}$$

(which gives a positive value of y) and where

$$\frac{\frac{4lp-1}{2}}{lq-m} < \frac{2p}{q} < \frac{\frac{4lp+1}{2}}{lq+m} \quad m = 2r+1 \text{ and } q + 4pm > 0. \quad (55)$$

Also

$$x = \frac{\frac{4lp-s-l}{2}}{lq-l \pm s}, \quad l = 4r-1, \; s = 4t+1$$

gives a positive y and its limit as $l \doteq \infty$ is

$$\frac{\frac{4p-1}{2}}{q-1},$$

for which y has a value $y = -|y|i$.

Similarly

$$x = \frac{4lp-s-l}{lq-l}, \quad l = 4r, \quad s = 4t+1,$$

gives a value of $y = -|y|i$, and its limit as $l \doteq \infty$ is

$$\frac{\frac{4p-1}{2}}{q-1}.$$

Now,

$$\overset{(y=-|y|i)}{\frac{\frac{4p-1}{2}}{q-1}} > \overset{(y'=+|y'|)}{\frac{\frac{4lp-s-l}{2}}{lq-l-s}} > \overset{(y''=+|y''|)}{\frac{\frac{4lp-s-l}{2}}{lq-l+s}} \quad \text{where} \quad \begin{aligned} l &= 4r-1, \\ s &= 4t+1, \\ q &> 4p, \end{aligned} \quad (56)$$

$$\frac{\frac{4l'p-s-l'}{2}}{l'q-l'} < \frac{\frac{4lp-s-l}{2}}{lq-l-s}, \quad \overset{(l'=4k)}{\frac{l'(q-4p)+s}{q-1}} < l,$$

hence

$$\begin{array}{ccccc} x & & x' & & x'' \\ (y=-|y|i) & & (y'=+|y'|) & & (y''=-|y''|i) \\ \dfrac{\dfrac{4p-1}{2}}{q-1} & > & \dfrac{\dfrac{4lp-s-l}{2}}{lq-l-s} & > & \dfrac{\dfrac{4l'p-s-l'}{2}}{l'q-l'}. \end{array} \tag{57}$$

As $l=\infty$, Lim $x''=x$ and however near x'' is to x, both giving values of y ($y=-|y|i$, $y''=-|y''|i$) purely negative imaginary, there is a value x', for which $y'=+|y'|$, and whose limit is also x, and $s=1, 5, 9, \cdots$. In like manner

$$\begin{array}{ccccc} x & & x' & & x'' \\ (y=+|y|) & & (y'=|y'|i) & & (y''=+|y''|) \\ \dfrac{2p}{q} & < & \dfrac{\dfrac{4lp+s}{2}}{lq+s} & < & \dfrac{2l'p}{l'q-s} \end{array} \quad \text{where} \quad \begin{array}{l} l=2r+1,\ l'=2k \\ s=4t+1 \end{array} \tag{58}$$

and

$$\frac{1}{4p}(l'(q-4p)-s)<l.$$

Hence however near the real positive point y'' is to the real positive point y by taking l' sufficiently large, an infinite number of purely imaginary points y' lie between by taking

$$l>\tfrac{1}{4}(l'(q-4p)-s).$$

Not only may l, l' be varied but $s=1, 5, 9, \cdots$. If x' is incommensurable there are infinitely many such points between x and x'' whose y's, y and y'', have positive real points. For

$$\begin{array}{ccccc} x & & x' & & x'' \\ \dfrac{2p}{q} & < & \dfrac{2p}{q}+\eta & < & \dfrac{2lp}{lq-s} \end{array} \quad \text{where} \quad \begin{array}{l} l=2r,\ s=2t+1 \\ \text{or } l=2r+1,\ s=2t, \end{array} \eta<\frac{2p}{q}\cdot\frac{s}{lq-s} \tag{59}$$

and where as η incommensurable $\doteq 0$, $y'=|y'|e^{u\pi i}$, lies between two positive points ($s=1, 3, 5 \cdots$ or $2, 4, 6 \cdots$) however near, as $l=\infty$, they may be.

IV. *The Real Curves Passing through the Discrete Real Points Between 0 and 1.*—Since for the real positive points $u=2m$, the real positive discrete points between 0 and 1 satisfy the equation $y=(x^{x-1}(1-x)^{x})^{1/1-2x}$ which has a real curve between 0 and 1, and discrete negative points between the same

limits and discrete positive and negative points elsewhere, and has spirals through its discrete and other real points, all of which spirals and multiple-valued points lie on the surface (25), and whose discrete points lie on the real parts of the curves $y = \pm(x^{x-1}(x-1)^x)^{1/1-2x}$. Similarly the negative discrete points between 0 and 1 of our function lie on the real part of the curve $y = -(x^{x-1}(1-x)^x)^{1/1-2x}$.

V. *The Spiral Systems.*

Since

$$\frac{x-1}{1-2x} + \frac{x}{1-2x} \equiv -1,$$

y between $-\infty$ and 0 can be expressed either as

$$y_1 = \frac{1}{x}\left(\frac{x-1}{x}\right)^{\frac{x}{1-2x}} = \frac{1}{\alpha}\left(\frac{\alpha+1}{\alpha}\right)^{\frac{x}{1-2x}} e^{v\pi i},$$

where

$$v = 1 + 2m\frac{x}{1-2x} \tag{60}$$

$(x = -\alpha)$, or as

$$y_2 = \frac{1}{x-1}\left(\frac{x}{x-1}\right)^{\frac{x-1}{1-2x}} = \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\frac{x-1}{1-2x}} e^{v'\pi i},$$

where

$$v' = 1 + 2m\frac{x-1}{1-2x}. \tag{61}$$

(See also (45).)

Since $v + v' = -2(m-1)$, $v'\pi = -2(m-1)\pi - v\pi$, and

$$y_1 = |y|e^{v\pi i} \text{ and } y_2 = |y|e^{-v\pi i} \text{ are conjugates.} \tag{62}$$

By § 3 y_1 and y_2 are families of right- and left-handed spirals respectively when $m > 0$.* The spirals are conjugate spirals which pierce the real plane and intersect in positive or negative real points. Symmetry shows the same thing between 1 and ∞.

* When $m < 0$ this is reversed. When $m > 0$, v increases with x, since $D_x v = \dfrac{2m}{(1-2x)^2}$.

Between 0 and 1 y can be expressed either as

$$y_1 = \frac{1}{x}\left(\frac{1-x}{x}\right)^{\frac{x}{1-2x}} e^{u\pi i}, \quad u = (2n+1)\frac{x}{1-2x} \tag{63}$$

or as

$$y_2 = \frac{1}{1-x}\left(\frac{x}{1-x}\right)^{\frac{x-1}{1-2x}} e^{u'\pi i}, \quad u' = 1 + (2n+1)\frac{x-1}{1-2x}. \tag{64}$$

As

$$u + u' = -2n, \quad e^{u'\pi i} = e^{-u\pi i},$$

hence

$$y_1 = |y|e^{u\pi i} \text{ and } y_2 = |y|e^{-u\pi i} \text{ are conjugates.} \tag{65}$$

y_1 and y_2 show y as families of right- and left-handed conjugate spirals, piercing the real plane and intersecting each other in real discrete points. ($n \geqq 0$. If $n < 0$ this is reversed.)

Thus from $-\infty$ to ∞ conjugate right- and left-handed spirals complete the representation and pierce the real plane in the discrete or other real points. The discrete complex points on the circle at x, finite or infinite in number as x is commensurable or incommensurable by (43), all lie in the surface $y = |y|e^{\phi i}$. As x advances we see how it is that the distribution of points on the circles presents an infinitely complex kaleidoscopic appearance and disappearance of points in the surface of revolution, but in such a way that the conjugate spirals are generated throughout. We consider especially the spirals between 0 and 1. The change in x for a change in u is

$$\Delta x = \frac{(1-2x)^2 \Delta u}{2n + 1 + 2(1-2x)\Delta u}, \tag{66}$$

since by (65) it is only necessary to consider one of the conjugate families, and furthermore we need only consider this one between 0 and $\frac{1}{2}$. If $\Delta u = 2$, the moving point will have made a complete turn on its spiral about OX. If $x = \frac{2}{5}(n = 0)$, and if $\Delta u = 2$, the change in x necessary for one complete turn and return of y to a positive real point is $\Delta x = \frac{2}{45}$. The point x arrived at is $x = \frac{2}{5} + \frac{2}{45} = \frac{4}{9}$, which is in fact seen to have a positive value for y.

$$\Delta u = \frac{(2n+1)\Delta x}{(1-2x)(1-2x-2\Delta x)}. \tag{67}$$

Between 0 and 1

$$x = \frac{u}{2n + 2u + 1}$$

and hence

$$x' = x + \Delta x = \frac{u + \Delta u}{2n + 2u + 1 + 2\,\Delta u} \tag{68}$$

gives points on the same spiral for a particular value of the parameter n. Let S_n denote a particular spiral of the family for a particular value of n. Corresponding to n is a point (x, y_n) which lies on S_n. When y is single-valued, (x, y) lies on every S_n or every S_n goes through (x, y) and corresponding to every value of n is (x, y). Thus when $x = 1/3$, y is single-valued and negative real, for

$$u = 2n + 1\,\frac{\frac{1}{3}}{1 - \frac{2}{3}} = 2n + 1,$$

and $e^{(2n+1)\pi i} = -1$, and $y = -6$. $(1/3, -6)$ belongs to every spiral of the family, that is, infinitely many conjugate spirals go through it. For $x = 1/5$, y is 3-valued, Fig. 2.

$$u = (2n + 1)\tfrac{1}{3}, \quad x = \frac{u}{2n + 2n + 1} = \frac{(2n + 1)\frac{1}{3}}{2n + 1 + 2(2n + 1)\frac{1}{3}}.$$

whence

$$\left.\begin{aligned}
x &= \underset{n=1}{\tfrac{1}{5}} = \underset{n=4}{\tfrac{3}{15}} = \underset{n=7}{\tfrac{5}{25}} = \text{etc.},\ u \equiv 1 \pmod 2\\
&= \underset{n=2}{\frac{\frac{5}{3}}{5+\frac{10}{3}}} = \underset{n=5}{\frac{\frac{11}{3}}{11+\frac{22}{3}}} = \underset{n=8}{\frac{\frac{17}{3}}{17+\frac{34}{3}}} = \text{etc.},\ u \equiv -\tfrac{1}{3} \pmod 2\\
&= \underset{n=3}{\frac{\frac{7}{3}}{7+\frac{14}{3}}} = \underset{n=6}{\frac{\frac{13}{3}}{13+\frac{26}{3}}} = \frac{\frac{19}{3}}{19+\frac{38}{3}} = \text{etc.},\ u \equiv \tfrac{1}{3} \pmod 2
\end{aligned}\right\} \tag{69}$$

In this case

$$\left.\begin{aligned}
&(\tfrac{1}{5}, -|y|) \text{ belongs to the spirals } S_{3n+1},\\
&(\tfrac{1}{5}, |y|e^{-(\pi/3)i}) \text{ belongs to the spirals } S_{3n+2},\\
&(\tfrac{1}{5}, |y|e^{(\pi/3)i}) \text{ belongs to the spirals } S_{3n}.
\end{aligned}\right\} \tag{70}$$

But

$$\begin{pmatrix}\text{The groups}\\ S_{3n},\ S_{3n+1},\ S_{3n+2}\end{pmatrix} = \begin{pmatrix}\text{The group}\\ S_n\end{pmatrix},\ n = 0, \pm 1, \pm 2, \cdots. \tag{71}$$

Hence at $x = 1/5$ *all the spirals* pass through the circle at $x = 1/5$ in three groups through 3 points on the circle.

For $x = \frac{1}{7}$, y is 5-valued.

$$\left.\begin{array}{lll}
(\frac{1}{7}, -|y|) & \begin{array}{l} n = 2, 7, 12, \cdots \\ u = 1, 3, 5, \cdots \end{array} & \text{belongs to } S_{5n+2} \\[1ex]
(\frac{1}{7}, |y|e^{3\pi i/5}) & \begin{array}{l} n = 1, 6, 11, \cdots \\ u = \frac{3}{5}, \frac{13}{5}, \frac{23}{5}, \cdots \end{array} & \text{belongs to } S_{5n+1} \\[1ex]
(\frac{1}{7}, |y|e^{\frac{7}{5}\pi i}) & \begin{array}{l} n = 3, 8, \cdots \\ u = \frac{7}{5}, \frac{17}{5}, \cdots \end{array} & \text{belongs to } S_{5n+3} \\[1ex]
(\frac{1}{7}, |y|e^{\frac{9}{5}\pi i}) & \begin{array}{l} n = 4, 9, \cdots \\ u = \frac{9}{5}, \frac{19}{5}, \cdots \end{array} & \text{belongs to } S_{5n+4} \\[1ex]
(\frac{1}{7}, |y|e^{\frac{1}{5}\pi i}) & \begin{array}{l} n = 0, 5, \cdots \\ u = \frac{1}{5}, \frac{11}{5}, \cdots \end{array} & \text{belongs to } S_{5n}
\end{array}\right\} \qquad (72)$$

But $(S_{5n}, S_{5n+1}, S_{5n+2}, S_{5n+3}, S_{5n+4})$

$$=(S_n), \quad n = 0, \pm 1, \pm 2, \cdots. \qquad (73)$$

Hence at $x = \frac{1}{7}$ *all the infinitely many spirals* pass in 5 groups through 5 points on the circle at $x = \frac{1}{7}$, whose radius is $|y|$. When x is incommensurable y has a different value for each value of n; thus y has an infinite number of values, and to each point (x, y_n) belongs only one spiral S_n. That is

$$\left.\begin{array}{l}
(x, y_0) \text{ belongs to } S_0, \\
(x, y_1) \text{ belongs to } S_1, \\
(x, y_2) \text{ belongs to } S_2, \\
\cdots\cdots\cdots \\
(x, y_{-1}) \text{ belongs to } S_{-1}, \\
(x, y_{-2}) \text{ belongs to } S_{-2}. \\
\cdots\cdots\cdots
\end{array}\right\} \qquad (74)$$

Here there is a one-to-one correspondence between the spiral and the point through which it passes. Through each point on the circle representing a value of y passes one and only one spiral. In the neighborhood of $x = \frac{1}{2}$, the spirals become infinitely compressed, since the number of turns becomes infinitely great as $x \doteq \frac{1}{2}$.

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THE RECONSTRUCTION OF THE MATHEMATICAL REQUIREMENT.

By Geo. W. Evans.

The following statement contains no material that has been decided upon by the Committee on the Curriculum in Algebra. I must assume personal responsibility for any mistakes of judgment.

In preparing this study of plans for the teaching of elementary mathematics, a careful survey has been made (1) of the report on foreign curricula made by the International Commission on the Teaching of Mathematics; (2) of the reports and discussions of the General Teaching Committee of the Mathematical Association of Great Britain; and (3) of reports and discussions in the United States. In general terms, this survey forces the conclusion that the constructive unrest so obvious in this country is not an isolated phenomenon, nor is a very radical reconstruction of our own plans likely to mark us as a peculiar people.

Except in Great Britain, the study of mathematics in the years corresponding to our seventh, eighth, and ninth grades has long been much more extensive, and at the same time much more concrete, than with us. Measurement and computation from models, the use of the pantograph in the study of similar figures, even the use of surveying instruments, is everywhere included in the program. Geometry and arithmetic are closely related, algebra and geometry are studied in the same years (in weekly or semi-weekly alternations, not "fused"). Applications of each subject are sought in the others. Approximate computation is included. Tables are used for interest and for similar matters. Logarithms are taught in the ninth grade or earlier. Continuous emphasis is laid on the function idea.

In Great Britain within fifteen years the actual text of Euclid has been disappearing from the schools. Examinations for the universities have been so liberally framed that it is now not only possible but indispensable to teach geometry as a science, not as an historic document. The most searching and stimulating book

in the world on the teaching of elementary algebra has been written by Dr. T. P. Nunn, and has been widely read, there and here. The subject of approximate computation as a school study has been created, and is now taught throughout England. Finally, in spite of the fact that English arithmetic is complicated by an antiquated and cumbrous monetary system, time is found to accomplish much more than our American schools at present can.

At the same time, and doubtless under the stress of war-time conditions, there is pressure for wider and more radical reform. A mathematician of world-wide authority* speaks of "the degeneration of algebra into gibberish," and goes on to say that the pupils "have got to be made to feel that they are studying something and not merely executing intellectual minuets."

Although what achievement there is belongs to the present century, the agitation in England goes back to mid-Victorian times, when the author of "Alice in Wonderland" wrote "Euclid and His Modern Rivals." There is a heartening echo from those early controversies in a letter from the Venerable J. M. Wilson, said to be the Nostradamus ridiculed by Dodgson. In this letter, dated in 1911, the veteran reformer reminds his successors that it is a syllabus for schools they are writing, and not a philosophic basis for algebra. He recommends in italics that the pupils be led to know mathematics as defined in the words of Comte, that is, as "the science of indirect measurement of magnitude, and the processes subsidiary thereto."

It must be confessed that some part of the reputed failure of mathematical teaching is to be attributed to the lack of unity commented on by the two personages last referred to. When we turn to the proposals for reform in the United States, taking as typical the Report on Algebra of the Missouri Society of Teachers of Mathematics and Science (1908), and the bulletin of the University of Texas on the Teaching of Geometry (1912), we find much excellent advice on the separate topics, lists of what we had better omit, suggestions as to approximate arithmetic, sources of problems, and so on, but no unifying general idea, no answer to the inevitable question, "What is it for?" To quote Whitehead again, "The pupils are bewildered by a

* A. N. Whitehead, in the *Mathematical Gazette*, January, 1916.

multiplicity of detail, without apparent relevance either to great ideas or to ordinary thoughts."

There is an interesting comment on the results of our American teaching from one of our French military visitors. He speaks of the student's loyauté d'esprit,—" no pose or touch of vanity, a disposition not to appear wiser than he really is lest he lose a chance of learning. *But,* there is an inordinate thirst for details, for separate definite facts, such as could be recorded in experience. This French officer had previously been a lecturer in literature at Johns Hopkins, and had found there the same tendency, the same " illusion de savoir," based on the accumulation of facts. He contrasts the education of the French student, who receives from the beginning an inheritance and tradition of general ideas. He is fed upon them, he loves them, at times he misuses them. In America, on the other hand, premature specialization predominates over general culture.

This difference appears clearly in a prolonged course of instruction. While with the French boy every bit of knowledge crystallizes about ideas already required, with the American new facts and ideas are fitted together like the stones in a mosaic, like the specimens in a museum. With the one, new knowledge is incorporated into a system already established, taking the place proper to its real importance; with the other, there is not always an effective criterion of structural place or relative importance.

He sums up his comment by the following statement, which might well be adopted as a pedagogical maxim in our secondary school mathematics:

" Surtout, n'enseignons des détails et des procédés pratiques qu'en les rattachant toujours aux quelques idées générales qui doivent être la base de notre enseignement."*

It is the "great ideas" that will give unity to our teaching; it is the "ordinary thoughts" that we seek in concrete applications. The untaught outsider considers mathematics remote and arid; if our pupils think so, wherewithal shall they be saved?

The United States is alone in hesitating to include numerical trigonometry in the syllabus of elementary mathematics. There

* André Morize, "Impressions d'un instructeur militaire francais," *Harvard Graduates' Magazine,* March, 1918.

is every reason why we should hesitate if this is simply to add another detail to our apparently crowded and heterogeneous list. But it obviously satisfies the requirement of concreteness, of connection with "ordinary thought." It is also an extension of the "great idea" of similarity. Will it crowd out other details of equal or greater importance? What shall be our criterion in choosing the items to retain? In other words, what do we mean by "important"?

There is general agreement that the high school course in mathematics "should be planned mainly for the students who never go to a university or college."* At the same time, since many of the students who take this course will go to college, it is certainly desirable to plan it so that another course can succeed it, giving without duplication the additional topics and training, if any, that will be required for admission to college. We may refer to these two courses as the "general mathematics course," for students who may or may not be going to college, and the "special mathematics course," or courses, for pupils who need further study in preparation for college, or for technical pursuits such as engineering, or commerce and finance.

This accords in the main with the aims of reforms demanded everywhere. They are well stated by Professor E. W. Hobson, F.R.S., in his presidential address before the British Mathematical Association in January, 1912. Democratization of Education is the formula he uses; and by it he means, not the extension of education to wider classes of the population, but rather the adaptation of educational methods to the *intellectual democracy;* that is, such a transformation of method and matter as to meet the needs of those who are lacking in exceptional capacity in relation to this particular subject. He gives a statement of the organization of subject matter, and the ideals underlying it, as it stood in England until within a few years. Without quoting it, I venture to say that many of our critics would find it an excellent basis for their animadversions.

What great idea can be used in the general mathematics course as the main intent of study? How shall we connect these great ideas with the ordinary thoughts of the pupil's life? The answer to these two questions will enable us to decide on the importance of details.

* Missouri Report, 1908, p. 2.

"The purpose of the first-year course should be to train pupils in the solution of problems by means of the equation, rather than to exercise them in abstract manipulations."*

"One of the main aims of the course in algebra is to develop the idea of functionality, and the various items of the syllabus should be treated with this end in view."†

"Algebra, well taught, helps to acquaint the pupil with the process of generalization, and to beget a clear consciousness of conditions under which the process may or may not be valid. . . . It introduces the idea of quantity changing continuously, and of the functional dependence of one quantity upon another. . . ."‡

"The central topic of algebra is, beyond question, the equation and its applications."§

"Applied problems, or, as they are often called, reading problems, form possibly one of the most important topics of elementary algebra."||

The one great idea is that of the functional relations; but it is out of the question to present that idea to the pupil except as the fruit of a considerable mathematical experience. It is not that the idea is in itself difficult; on the contrary, it is so simple that the pupil is likely to see in it nothing fruitful. Like all abstractions that have come late to the mind of humanity, it cannot be stated as the basis for youthful study, but must come as a generalization when there is sufficient material for its induction, and when it need not remain inert for lack of concepts that it may serve to illuminate. All mathematical instructions must bear towards this as a goal. The teacher must have it in mind, and the pupil's advancement must be consciously directed along lines that will present first one and then another of the details upon which finally this great generalization can be based.

Another idea, a "great idea," to be sure, and not quite so remote as the function idea, is that of generalization. That, also, is to be in the mind of the teacher, but would surely be meaningless to the pupil until after a good deal of algebraic work. A formula is a generalized solution. Yes, but the pupil will

* Central Association Report, 1907, p. 3.

† British Report on Mathematics for Girls, 1916, p. 6.

‡ British Report on Algebra and Trigonometry, 1911, pp. 2 and 3.

§ "The Teaching of Mathematics," J. W. A. Young, 1907, p. 302.

|| Schultze, "Teaching of Mathematics," p. 330.

think it a rule for arithmetical application rather than a pattern solution. He will come to the generalization point of view, later; but he will have seen it himself, then. Sudden revelations dazzle him.

The principle of substitution is only a restricted aspect of generalization.

There remains the equation. So far as school algebra is concerned, it is a symbolic statement of rather complicated numerical facts, which can be systematically transformed into one or more direct statements of numerical value. On the first day of this algebra work, the pupil confronts a sequence of equations offered to him as abbreviations of the successive steps of a verbal argument. The argument in its unabbreviated form must be intelligible to him, otherwise, of course, the symbols will not be. That is to say, the algebra is unnecessary. The problem can be done "by arithmetic." What is it for then? Only to get accustomed to the symbolism, so that the more difficult problems can afterwards be "explained" symbolically without being unintelligible. It is a generalization of method.

By means of problems suitably chosen, the equation can be followed through successive complications until we have completed its theory so far as elementary algebra is concerned. We have only to find the problems, and that we shall refer to later. A number of processes of transformation will have to be studied, such as the "four operations" for algebraic expressions, and so on. These topics can remain subsidiary, and need not be enlarged into exhaustive treatises. That restriction will exclude a great deal of manipulative work, which all our critics, even the friendly ones, agree upon as the main obstacle to intelligent progress as distinguished from the acquisition of mechanical facility.

Incidentally the equation compels the teaching of other things, —subsidiary, but of fundamental importance. For one, negative numbers. You have to deal with them when they appear as values of the unknown letter. You can postpone the evil day by saying that the problem was stated in error, but the natural impulse to generalization, which algebra fosters, will eventually force the issue. With quadratics it is unavoidable. There is one false start which should always be avoided, but almost never

is. That is the equation whose solution is positive, but on account of a foolish rule of transposition presents a negative number on one side (the right side, if you please), and x with a negative coefficient on the other. That needless perplexity should always be avoided by adding enough to each side to get rid of all negative terms, and by putting confidence in the perfect symmetry of the equation sign. It isn't new, for Mohammed Ben Musa did it; so did Diophantus.*

Conceding that the equation serves very well as a unifying topic for algebra, let us consider what would be of corresponding worth in geometry. There is, in the first place, the notion of congruence, derived from our experiences with material things; if a machine fits together here, it will also fit together wherever we carry it,—even in an aëroplane miles above the earth, or in a mine, or in any land, however distant. What is the easiest way of deciding, by measurement, when material solids will fit together? Again, we see about us many structures which hold shape permanently. Can our geometry decide what parts in such a structure need to be devised to hold it together? Compare the jointed parallelogram and the jointed triangle. One is not permanent in form, the other is. How decide?

Again, there is the notion of similarity, upon which are based all maps and plans and models. In making a reduced copy of a plan, if all distances are drawn to scale, the angles in the new plan will all be the same as in the original. If we make all angles in the new plan the same as in the original, all the distances will be drawn to scale. If we construct a model of a tank, say to the scale of one eighth of an inch to a foot, the depth of the tank will be 96 times as great as the depth of the model, but the capacity of the tank will be 884,736 times as great as the capacity of the model.

The elementary course in geometry will be a scientific study of these two notions. The general aim will be to secure convenient and well-reasoned methods of indirect measurement.

Since this scientific study involves numerical statements more or less complicated, there will arise occasion for the use of algebraic equations, not only in the discussion of these methods of measurement, but also in problems arising from them.

* T. L. Heath, "Diophantus of Alexandria," 185, p. 89.

Accompanying all this work, in algebra as well as in geometry, should be a careful study of the accuracy attainable from the numerical data available. From this study the pupil not only obtains much needed practice in computation, but also constant occasion for self-reliant judgment and for a critical attitude towards his own results. Systematic methods of checking all computations should be carefully reasoned out for these approximate methods, and should be rigorously used in all cases. Checking our results should not be regarded as a minor matter to be resorted to only when our computation for the check is comparatively easy; on the contrary, if accuracy can only be assured by independent computation fully as difficult as the original work, it should be carried through without fail. Surveyors are often required to swear to their results in courts of law; navigators risk valuable property, and even human lives, on their accuracy of computation; tall buildings and bridges depend for their stability not only on the solidity of their foundations, but quite as much on the hundreds, and even thousands of separate computations by which the details of these structures are determined.

We have then these four topics which we may use in unifying the general course in mathematics: (1) Approximate computation, (2) the equation as a means of solving problems, (3) congruence, (4) similarity.

We shall find, I think, that most of the content of elementary algebra and of plane geometry would be included, except the very things that are most criticised on account of their formalism. Certain other details of a rather complicated character, such as the binomial theorem and geometric series, though undoubtedly useful and important, would have to be postponed as too technical for this introductory course. The complete schedulization of available topics is a matter of detail that will have to be left for a later day, and for more well-considered and widely discussed formulation.

Charlestown High School,
Boston, Mass.

A GEOMETRIC ILLUSTRATION OF LIMITS.

By Carl Eben Stromquist.

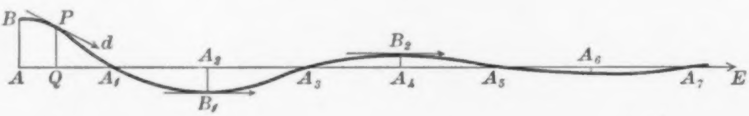
Teachers of freshman mathematics, the calculus, or review courses for high school teachers often experience some difficulty when the topic of limits comes up for discussion. The difficulty arises from the fact that the teacher must first "unteach" an erroneous definition of limits which is still to be found in many of our high school texts. The following definition, taken from a rather widely used plane geometry text, is typical of the error. "The limit of a variable is a constant from which the variable can be made to differ by less than any assigned quantity, but to which it can never be made equal." In a random selection of ten texts five gave essentially this definition. One of these was published as recently as 1915. The mischief is caused by the last part of the statement, viz., "but to which it can never be made equal."

Just why this part ever was or at present is included in the definition is difficult to explain, unless it is because in practically all the theorems and problems where the definition is needed and that the student meets in his high school mathematics the variable does not reach the limit. Thus, the perimeter, or area, of a regular polygon inscribed in a circle never reaches the circumference, or area, of the circle. As a matter of fact, geometric illustrations of cases where the variable does reach the limit are almost entirely lacking in our elementary texts. It is the purpose of this article to present an illustration which the writer has found helpful in his classes.

Before giving the illustration it may be well to state a definition of limit which is free from the objection already raised. The following will suffice for elementary purposes at least. If a variable x represents any one of an infinite series of values, it is said to approach a constant quantity a as a limit provided that the numerical value of the difference $x - a$ finally becomes and remains less than any preassigned positive quantity. It is

important to observe the following three conditions which are involved in this definition, viz., that x must vary according to some law, that the difference $x-a$ must become numerically less than any assigned number, and, thirdly, that as x continues to vary the difference $x-a$ must remain less than this number.

To illustrate how a variable may satisfy the preceding definition of a limit and still reach the limit, imagine a race track constructed according to the following specifications. A line AE is drawn in an easterly direction from a given point A. On this line equal segments, say 100 yards, AA_1, A_1A_2, A_2A_3, etc., are laid off, and at A, A_2, A_4, etc., perpendiculars AB, A_2B_1, A_4B_2, etc., are laid off alternately to the north and to the south of the line AE, and such that each of these perpendiculars is equal to one half of the preceding one. The race track is now to be constructed by drawing a smooth curve through the points B, A_1, B_1, A_3, B_2, A_5, etc., as in the figure below. The direction d



at which a horse will be running at any point P of the race track will be the direction of the tangent line drawn to the curve at this point. This direction will vary for different positions of P, but it is evident that, as P moves further and further away from B, the variable d will approach more and more an easterly direction, *i. e.*, a direction parallel to the line AE. It is evident, intuitively, that the definition of a limit is here satisfied. The variable d, however, reaches the limit at all the points B, B_1, B_2, etc.

Attention might also be called to the fact that, if a perpendicular PQ be drawn from P to AE, this perpendicular approaches zero as a limit as P moves away from B. Here again the variable PQ reaches its limit, zero, at the points A_1, A_3, A_5, etc.

The race track as here described depends upon a free hand construction. A more accurate construction may be obtained by drawing the graphs of certain equations which the student may have occasion to study later, such as, $y=\sin x/x$, or $y=e^{-x}\sin x$.

University of Wyoming,
Laramie, Wyoming.

CHARACTER-BUILDING CONTENT OF ARITHMETIC.

BY JENNY LIND GREEN.

Arithmetic was introduced into American schools to meet definite social situations of Colonial times. Although a fixed factor in the course of study, it has wandered far from its predestined course. Literally, we have seemed to stress addition, subtraction, multiplication and division at the expense of knowing what to add, subtract, multiply and divide. We have given the impression of emphasizing the fundamentals and their kindred abstractions to the extent of ignoring the fact that it is sometimes necessary for us to decide which of these desirable processes should be used. Sadder still, we have been accused of forgetting to mention in a functional way that one is likely to meet these fundamentals and mathematical accessories in the flesh, and clothed in raiment the like of which was never seen in text-book or classroom.

We recognize such criticisms as following in the wake of definite social interpretations of character-building.

Education, February, 1916, Ernest P. Carr: "Dr. Nicholas Butler says, 'That knowledge is of most worth which stands in closest relation to the higher forms of activity of that spirit which is created in the image of Him who holds the world in the hollow of His hand.' The world needs to have its eyes opened to live and not exist. The school must open its eyes."

The social meaning of character-building has characterized educational writing of the last decade. Judged by its standards a curriculum is character-building in content to the extent that it presents social situations with regard to the development of appreciation of the best work of the world and ability and inspiration to help it along.

The social meaning has revived and enriched the original values of arithmetic. Articles written during the last few years dealing with subject-matter and method of arithmetic have been

colored with the problem of how the subject can help children live. Studies published during the last two years show that attitudes and practices in our schools are more in harmony with social need. New texts afford striking evidence that arithmetic content is beginning to meet social demands.

The change in texts deserves more than mere mention. Publishers ordinarily avoid responsibility for texts not demanded by the public. They are not good investments. That a number of arithmetic texts of the type mentioned are on the market indicates that we may be somewhat assured of the beginnings of firmer foundations for carrying out a socialized program.

The extent to which the social situation has been made a determining factor in texts is shown in terminology, organization and problem content. A glance at chapter organization will suggest it.

Text typical of ten years ago:

Chapter organization:

I. Processes with Integers.
II. Common Fractions.
III. Decimal Fractions.
IV. Denominate Numbers.
V. The Solutions of Problems.
VI. Percentage.
VII. Business Applications.
VIII. Interest and Banking.
IX. Powers and Roots.
X. Mensuration.

Text published recently for use in the same grades:

Junior High School Mathematics—Wentworth, Smith, Brown.

Book I.

Chapter organization (only arithmetical phases are quoted):

I. Arithmetic of the Home.
II. Arithmetic of the Store.
III. Arithmetic of the Farm.
IV. Arithmetic of Industry.
V. Arithmetic of the Bank.

Book II.

Chapter organization (only arithmetical phases are quoted):

I. Arithmetic of Trade.
II. Arithmetic of Transportation.
III. Arithmetic of Industry.
IV. Arithmetic of Building.
V. Arithmetic of Banking.
VI. Arithmetic of Corporations.
VII. Arithmetic of Home Life.
VIII. Arithmetic of Farming.
IX. Arithmetic of Community Life.
X. Arithmetic of Civic Life.
XI. Arithmetic of Investments.
XII. Arithmetic of Mensuration.

Terminology and organization of the older texts were determined by the abstract process. They contained some concrete problems. These however were few and far between and were not as a series cumulative to definite social ends appreciated by the pupil. The terminology and organization of the new text are largely determined by social needs. Problem content and its organization in the several chapters show the same influence.

The new texts stress three types of training:

1. Training to see social situations involving arithmetical relations.
2. Training to know what processes are essential to the arithmetical phase of social situations.
3. Training to use those processes economically.

All texts have stressed some of these values. Those published recently recognize the social situation as the determining factor.

Judging from such indications arithmetic is ridding herself of the barnacles accumulated on her voyages and is taking unto herself the general social content necessary to right living.

The content of arithmetic is functional to the extent that the arithmetical elements are presented in the natural settings. One rarely, if ever, meets as a social problem the necessity for combining 2 and 2. One may find it necessary to add 2 dollars and 2 dollars or 2 apples and 2 more apples. One is much more likely to meet such a need in combination with a number of other relations which add to its social significance.

The social situations of arithmetic sometimes involve ethical elements. The relation is often close. The arithmetic of trade has an inseparable companion in the ethics of trade. The arithmetic of trade had its development determined somewhat by ethical principles. The history of measurement affords illustration. The ethics of keeping accounts, paying bills promptly, etc., bear closer relation to arithmetic than to any other subject-matter in that they occur together oftener in the same social situation. The ethics of thrift is not separate in conduct from the arithmetic of thrift. There are certain types of ethical principles which are a part of social situations of arithmetic.

These ethical principles are important. The stress of moral problems to which we are being rudely awakened in the present international crisis forces to attend to them. It asks us such questions as these: Is it of great value to the world that individuals should know how to calculate their economic benefits of trade without an appreciation of how that trade affects the good of the mass? Is it of particular value that our children are able to balance accounts if they do not have that ethical appreciation which should govern their expenditures? In general, is it good that present and succeeding generations should be trained to know what to do in meeting the arithmetic of situations and not be trained in the right conduct involved?

A forecast of a type of ethical values of arithmetic is seen in recent discussions of thrift.

Child Welfare Bulletin, 1918: "Thrift as a matter of self-preservation should be taught in school and at home. Its value not only economically but morally is of the greatest moment."

N. E. A. Bulletin, 1917: "Thrift is a patriotic duty. The nation of to-day is learning the economic necessity of thrift, but the nation of to-morrow must know the educational necessity of this virtue."

Such quotations are typical of many made by both laymen and educators. A supplementary arithmetic suggestive of ethical relations of this special type is that of Superintendent Farmer, of Evanston, Illinois. It is published by Ginn & Co. It emphasizes food conservation.

The public is awake to a pressing need for the kind of ethical training arithmetic might afford. That it should be given in

connection with arithmetic is evident in that its principles are more closely related to the subject-matter of arithmetic than to any other special subject-matter. If such ethics should be a part of any special subject it should be a part of arithmetic.

The open question is, should it be a part of any special subject, or should it be apart from all other types of relations?

As a matter of fact nearly every conceivable plan for teaching ethics is being tried at present—a frank confession of past failures. There was a time when we felt that history might solve such of its problems as needed to be taught in school. A study of state courses shows that the study of civics is expected to function definitely in that respect. This is evident in the statements of aims.

A questionnaire to which twenty-two of our large cities responded shows fifteen statements of aims attaching moral significance to civics study.

Entire list of cities: Memphis, Pittsburgh, Seattle, Carson City, Philadelphia, Boise, St. Paul, Birmingham, Little Rock, Chicago, Syracuse, New York City, Washington, D. C., Richmond, Spokane, Nashville, Columbus, Los Angeles, Cincinnati, Jacksonville, Baltimore, Santa Fe.

Fourteen of the group of fifteen believed that some civic material should be taught in connection with the subjects to which it is most closely related. Statements of practices in correlation were given. These practices varied widely. They ranged from correlations with literature only to correlations with all of the usual elementary-school subjects with the exception of arithmetic.

Ten of the group stated that some practical work was done with local community problems.

Such attitudes and practices seem to indicate the following:

1. That ethical training of the past has not proved satisfactory.

2. That ethical values are still linked somewhat with the study of civics.

3. That there is a decided tendency to break up this civics unit as a separate unit and use it in connection with subject-matter closely related to it.

That the tendency to correlate material has not already carried civics over into arithmetic is partly explained in the fact

that civics courses have not, as a rule, included the ethics of arithmetic. Society has not demanded it.

Society has demanded an education that prepares for life. One response to the demand has been a more socialized content of arithmetic. To-day society is conscious of a type of ethical appreciation essential to her welfare, which the school has not given. This ethics belongs to arithmetic. Past experiences and present tendencies in ethics teaching show that the road is "open."

UNIVERSITY ELEMENTARY SCHOOL,
CHICAGO UNIVERSITY.

NEW BOOKS.

Introduction to the Elementary Functions. By RAYMOND BENEDICT McCLENON; edited by WILLIAM JAMES RUSK. Boston: Ginn and Company. Pp. ix + 244. Price $1.80.

This book, for the use of freshmen in colleges and technical schools and of advanced pupils in secondary schools, presents a unified course, the advantages of which are claimed to be: (1) It avoids duplication. (2) It connects better with secondary-school matematics than does either solid geometry, college algebra, or trigonometry. (3) It gives a more adequate idea of mathematics in content and methods, because the fundamentally important idea of functionality is brought into the central position from the beginning. (4) It develops greater resourcefulness in the student, encouraging him to use methods already mastered.

The material includes the most important topics usually treated under trigonometry and elementary analytic geometry, with a simple introduction to the differential calculus. The method of presentation is inductive. Clearness of explanation and demonstration has been aimed at and brevity achieved by omitting those steps which the student himself can supply.

NOTES AND NEWS.

THE WHITE HOUSE

WASHINGTON, July 31, 1918.

My dear Mr. Secretary:

I am pleased to know that despite the unusual burdens imposed upon our people by the war they have maintained their schools and other agencies of education so nearly at their normal efficiency. That this should be continued throughout the war and that, in so far as the draft law will permit, there should be no falling off in attendance in elementary schools, high schools or colleges is a matter of the very greatest importance, affecting both our strength in war and our national welfare and efficiency when the war is over. So long as the war continues there will be constant need of very large numbers of men and women of the highest and most thorough training for war service in many lines. After the war there will be urgent need not only for trained leadership in all lines of industrial, commercial, social and civic life, but for a very high average of intelligence and preparation on the part of all the people. I would therefore urge that the people continue to give generous support to their schools of all grades and that the schools adjust themselves as wisely as possible to the new conditions to the end that no boy or girl shall have less opportunity for education because of the war and that the Nation may be strengthened as it can only be through the right education of all its people. I approve most heartily your plans for making through the Bureau of Education a comprehensive campaign for the support of the schools and for the maintenance of attendance upon them, and trust that you may have the coöperation in this work of the American Council of Education.

Cordially and sincerely yours,

WOODROW WILSON.

HON. FRANKLIN K. LANE,
Secretary of the Interior.

Education in Patriotism.—Education in patriotism, and the agencies, official and unofficial, engaged in promoting patriotic work in the schools are reviewed in Teachers' Leaflet No. 2, just issued by the Bureau of Education of the Department of the Interior.

The work of the Council of National Defense, National Security League, National Committee of Patriotic Societies, National Board for Historical Service, The Bureau of Education, National Education Association, and Committee on Public Information, is described in some detail so that teachers and school officers may have ready at hand abundant sources of help in spreading the teaching of true Americanism.

"It is the contribution of American schools, and particularly of colleges and universities," says the leaflet, "to further the teachings of patriotism in the present emergency, and the opportunity of rendering such service has everywhere been eagerly accepted. Since, however, a number of organizations of national scope, some governmental, others privately supported, are now endeavoring to work through the schools of the country in the cause of education in patriotism, considerable confusion has arisen in the minds of school men regarding the origin and purpose of the various agencies at work. This leaflet aims to put them in touch with the material available and to describe the work of the leading organizations already in the field."

Copies of "Education in Patrioism" will be sent on application to the Commissioner of Education, Washington, D. C.

New Members.

John C. Bechtel, Germantown High School, Philadelphia, Pa.
Harriet W. Sheppard, 31 Westview St., Mt. Airy, Pa.
Floyd B. Watson, Rockville Center, Long Island, N. Y.
Mary M. Kemp, LeRoy, N. Y.